Maximum Likelihood Decoding Analysis of Accumulate-Repeat-Accumulate Codes

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Abstract

Repeat-Accumulate (RA) codes are the simplest turbo-like codes that achieve good performance. However, they cannot compete with Turbo codes or low-density parity-check codes (LDPC) as far as performance is concerned. The Accumulate-Repeat-Accumulate (ARA) codes, as a subclass of LDPC codes, are obtained by adding a pre-coder in front of RA codes with puncturing, where an accumulator is chosen as a precoder. These codes not only are very simple, but also achieve excellent performance with iterative decoding. In this paper, the performance of these codes with (ML) decoding are analyzed and compared to random codes by very tight bounds. The weight distribution of some simple ARA codes is obtained, and through existing tightest bounds we have shown the ML SNR threshold of ARA codes approaches very closely to the performance of random codes. We have shown that the use of precoder improves the SNR threshold but interleaving gain remains unchanged with respect to RA code with puncturing.

Topic areas: Information Theory, Coding Theory and Practice

Key words: Performance analysis, weight distribution, upper bounds, asymptotic thresholds, random codes, LDPC codes, maximum likelihood decoding.

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I. INTRODUCTION

Low Density Parity Check (LDPC) codes were proposed by Gallager [1] in 1962. After introduction of turbo codes by Berrou et al [2] in 1993, researchers revisited the LDPC codes, and extended the work of Gallager. After Gallager huge number of contributions have been made to LDPC codes see for example [3], [6], [9], [10], [11], [12] and references there.

The advent of Turbo codes; introduced in [2]; has started a big movement towards the invention of a myriad of new code structures. The basic properties of these codes are the ability of iterative decoding and using pseudorandom interleavers. A pretty wide class of code structures called Turbo-like codes was introduced in [4]. Repeat-Accumulate codes (RA) are perhaps the simplest codes among this class. Simplicity of these codes lends itself to a more comprehensive analysis of their performance. Divsalar et al. [4] have shown that the performance of these codes with ML decoding can achieve near Shannon-limit performance. Moreover, they have proven that it achieves the Shannon-limit when rate goes to zero. Irregular Repeat-Accumulate (IRA) codes could achieve much better performance, which was shown by Jin et al. [5]. Jin presented a method for designing very good IRA codes for binary erasure and additive white Gaussian channels. He showed that they outperform Turbo codes for very large block sizes. In this paper first we analyze the performance of RA codes with regular puncturing. We show that with increasing the repetition and puncturing the output of the accumulator we can construct better codes for rate 1/2 as far as the performance is concerned. Furthermore, we show that the use of an accumulator as a precoder further improves the ML decoding performance.

II. MAXIMUM LIKELIHOOD DECODING ANALYSIS

Since there is no practical ML decoding algorithm available for block codes with large block size, we use the performance bounds to obtain some insight on codes behavior. In [7] Divsalar (see also [8]) provides a tight upper bound on frame (word) error rate (FER) and bit error rate (BER) for a (n, k) linear block code with code rate $R_c = k/n$ and distance spectrum A_d (number of codewords with weight d), decoded by Maximum Likelihood criterion over an additive white Gaussian noise (AWGN) channel. It also provides a minimum Eb/No threshold with closed form expression. We use this bound throughout the paper. Define the normalized distance (with respect to all zero codeword) as $\delta = d/n$, and the normalized distance spectrum which is also called the rate distance spectrum as $r(\delta) = \frac{\ln A_d}{n}$, then the FER bound in [7] or [8] can be expressed as:

$$P_e \leq \sum_{d=d_{min}}^{d_{max}} e^{-nE(\delta,\, \beta\, \frac{E_c}{N_o})} \tag{1}$$

where

$$E(\delta,\, \beta\, \frac{E_c}{N_o}) = -\frac{1}{2}\ln(1-\beta+\beta e^{-2r(\delta)}) + \frac{\beta\delta}{1-(1-\beta)\delta}\,\frac{E_c}{N_o} \tag{2}$$

and

$$\beta = \frac{1-\delta}{\delta}\left[\sqrt{\frac{E_c}{N_o}\frac{\delta}{1-\delta}\frac{2}{1-e^{-2r(\delta)}} + \left(1+\frac{E_c}{N_o}\right)^2 - 1} - \left(1+\frac{E_c}{N_o}\right)\right] \tag{3}$$

where $\frac{E_c}{N_o} = R_c\frac{E_b}{N_o}$, and $0 < \beta < 1$. When $\beta = 1$, the bound reduces to union bound. To compute BER bound just replace A_d with $\sum_{w=1}^{R_c n} \frac{w}{R_c n} A_{w,d}$ in the FER bound where $A_{w,d}$ is the number of codewords with input weight w, and output weight d. An important result of this bound is the tightest closed-form threshold on minimum E_b/N_o that can be written as

$$\left(\frac{E_b}{N_o}\right)_{min} \leq \frac{1}{R_c}\max_{\delta}(1-e^{-2r(\delta)})\frac{1-\delta}{2\delta} \tag{4}$$

III. ANALYSIS OF RA CODES WITH PUNCTURING

Repeat-Accumulate codes are the simplest codes among Turbo-like codes, which make them very attractive for analysis. In RA code an information block of length N is repeated q times and interleaved to make a block of size qN, and then followed by a rate-1 accumulator (see Fig. 1). We use

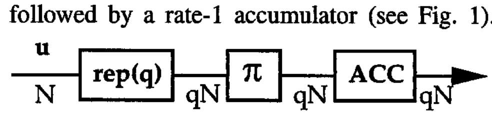

Fig. 1. Repeat-Accumulator code block diagram

the concept of uniform interleaver [13] to compute the overall input-output weight enumerator (IOWE). In this paper the final derived IOWE for concatenated codes should be considered as the averaged IOWE over all interleavers between repetition code and inner punctured accumulator. Therefore, we need to compute the IOWE of both repetition code and the accumulator. For repetition code it is simply the following:

$$A_{w,d}^{rep(q)} = \binom{N}{w}\delta_{d,qw} \tag{5}$$

where $\delta_{i,j}$ is Kronecker delta function. The IOWE of the accumulator is:

$$A^{acc}_{w,d} = \binom{N-d}{\lfloor \frac{w}{2} \rfloor}\binom{d-1}{\lceil \frac{w}{2} \rceil - 1} \tag{6}$$

To compute the IOWE of the RA codes with puncturing we use the equivalent graph depicted in Fig. 2 instead of the accumulator with puncturing. Puncturing uses a periodic pattern $0 \ldots 0X$ with period p, where zeros indicate the puncturing positions. As we see the equivalent code, to compute IOWE, is a concatenated code of a regular check code and an accumulator, which is shown in Fig. 3.

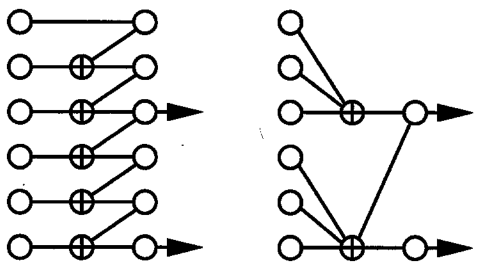

Fig. 2. Accumulator with puncturing and its equivalent graph for p=3.

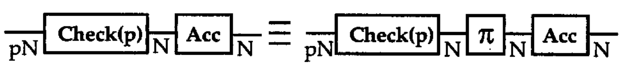

Fig. 3. Block diagram of accumulator with puncturing

Since the check code is regular and memoryless, the presence of any interleaver between two codes does not change the IOWE of the overall code. In order to compute the IOWE for this code we insert a uniform interleaver between two codes. The next step is to compute the IOWE of the check code. The IOWE of check code can be expressed in a simple closed form formula if we use the two dimensional Z-transform of IOWE denoted by $A^c(W, D)$. The inverse Z-transform results in $A^c_{w,d}$. We start with N=1, i.e. we have only one parity-check. We have

$$A^c(W, D) = E_p(W) + O_p(W)D \tag{7}$$

where $E_p = Even[(1 + W)^p]$, and $O_p = Odd[(1 + W)^p]$. Since there are N independent check nodes in the code, the IOWE can be written in Z-transform as:

$$\begin{aligned} A^c(W, D) &= [E_p(W) + O_p(W)D]^N \\ &= \sum_{d=0}^{N} \binom{N}{d} [E_p(W)]^{N-d}[O_p(W)]^d D^d \end{aligned} \tag{8}$$

The IOWE is obtained by taking the inverse Z-transform. The closed form expression for $A^c_{w,d}$ for arbitrary p can be derived but it is very complicated. Instead in this paper we derive the IOWE for p=2, 3, and 4, which are practically more useful.

A. Case p = 2

Using the general formula in Z-transform we have:

$$A^{c(2)}(W, D) = (1 + W^2 + 2WD)^N \tag{9}$$

It can be expanded as:

$$\begin{aligned} A^{c(2)}(W, D) &= \sum_{d=0}^{N} \sum_{w=0}^{2N} \binom{N}{d} \sum_{j=0}^{N-d} \binom{N-d}{j} \\ &\quad \times 2^d \delta_{w,2j+d} W^w D^d \end{aligned} \tag{10}$$

Therefore the IOWE can be expressed as

$$A^{c(2)}_{w,d} = \binom{N}{d} \sum_{j=0}^{N-d} \binom{N-d}{j} 2^d \delta_{w,2j+d} \tag{11}$$

B. Case p = 3

Starting from general formula in Z-transform we have:

$$A^{c(3)}(W, D) = [1 + 3W^2 + (3W + W^3)D]^N \tag{12}$$

It can be expanded as:

$$\begin{aligned} A^{c(3)}(W, D) &= \sum_{d=0}^{N} \binom{N}{d} \sum_{ii=0}^{N-d} \binom{N-d}{ii} 3^{ii} W^{2ii} \\ &\quad \times \sum_{i=0}^{d} \binom{d}{i} 3^{(d-i)} W^{2i} W^d D^d \end{aligned} \tag{13}$$

Let $i + ii = j$, then it is easy to show that

$$\begin{aligned} A^{c(3)}(W, D) &= \sum_{d=0}^{N} \binom{N}{d} \sum_{j=0}^{N} \sum_{i=\max(0,j-N+d)}^{\min(j,d)} \binom{d}{i} \\ &\quad \binom{N-d}{j-i} 3^{(d+j-2i)} W^{(d+2j)} D^d \end{aligned} \tag{14}$$

Therefore the IOWE is:

$$\begin{aligned} A^{c(3)}_{w,d} &= \binom{N}{d} \sum_{j=0}^{N} \sum_{i=\max(0,j-N+d)}^{\min(j,d)} \binom{d}{i} \binom{N-d}{j-i} \\ &\quad 3^{d+j-2i} \delta_{w,2j+d} \end{aligned} \tag{15}$$

C. Case p = 4

The code for this case can be viewed as a concatenated code as shown in Fig. 4. Because the check code is regular and memoryless, we can put any interleaver between the codes without changing the IOWE of the overall code. By using a

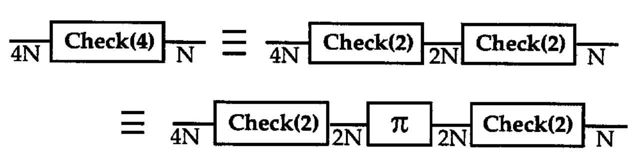

Fig. 4. Block diagram of check 4 code and its equivalents

uniform interleaver and the results found for case p=2 the IOWE can be written as:

$$A_{w,d}^{c(4)} = \sum_{h=0}^{2N} \frac{A_{w,h}^{c(2)} A_{h,d}^{c(2)}}{\binom{2N}{h}} \tag{16}$$

Using the result for $A_{w,d}^{c(2)}$, we obtain

$$A_{w,d}^{c(4)} = \binom{N}{d} \sum_{j=0}^{N-d} \sum_{i=0}^{2N-d-2j} \binom{2N-d-2j}{i} \binom{N-d}{j} 2^{2d+2j} \delta_{w,d+2i+2j} \tag{17}$$

This method can be applied for any p that can be decomposed into two smaller numbers. Having computed the IOWE of the check code, we can use the uniform interleaver formula to compute the IOWE of the accumulator with puncturing. We have

$$A_{w,d}^{acc(p)} = \sum_{h=0}^{N} \frac{A_{w,h}^{c(p)} A_{h,d}^{acc}}{\binom{N}{h}} \tag{18}$$

$$A_{w,d}^{acc(3)} = \sum_{h=0}^{N} \sum_{j=0}^{N} \sum_{i=\max(0,j-N+h)}^{\min(j,h)} \binom{h}{i} \binom{N-h}{j-i} \binom{N-d}{\lfloor \frac{h}{2} \rfloor} \binom{d-1}{\lceil \frac{h}{2} \rceil - 1} 3^{h+j-2i} \delta_{w,2j+h} \tag{19}$$

and

$$A_{w,d}^{acc(4)} = \sum_{h=0}^{N} \sum_{j=0}^{N-h} \sum_{i=0}^{2N-h-2j} \binom{2N-h-2j}{i} \binom{N-h}{j} \binom{N-d}{\lfloor \frac{h}{2} \rfloor} \binom{d-1}{\lceil \frac{h}{2} \rceil - 1} 2^{2h+2j} \delta_{w,h+2i+2j} \tag{20}$$

It should be noted that despite the fact that we use a uniform interleaver to obtain the IOWE, we come up with the exact IOWE for accumulator with puncturing. The next step is to find the IOWE of the systematic RA code with puncturing, which is derived in case of a uniform interleaver after repetition as

$$A_{w,d}^{rep(q)-acc(p)} = \sum_{l=0}^{qN} \frac{A_{w,l}^{rep(q)} A_{l,d-w}^{acc(p)}}{\binom{qN}{l}} \tag{21}$$

For systematic punctured RA (q=3,p=3)

$$A_{w,d}^{rep(3)-acc(3)} = \frac{\binom{N}{w}}{\binom{3N}{3w}} \sum_{h=0}^{N} \sum_{j=0}^{N} \sum_{i=\max(0,j-N+h)}^{\min(j,h)} \binom{h}{i} \binom{N-h}{j-i} \binom{N-d+w}{\lfloor h/2 \rfloor} \binom{d-w-1}{\lceil h/2 \rceil - 1} 3^{h+j-2i} \delta_{3w,2j+h} \tag{22}$$

Now we obtain the asymptotic expression of $r(\delta)$ for punctured RA (q=3,p=3), after summing (22) over w. Let $\delta \triangleq \frac{d}{2N}$ for $0 < \delta < 1$, $\eta \triangleq \frac{h}{2N}$ for $0 < \eta < 1/2$, $\rho_1 \triangleq \frac{i}{2N}$ for $\max(0, \rho_2 + \eta - \frac{1}{2}) < \rho_1 < \min(\rho_2, \eta)$, and $\rho_2 \triangleq \frac{j}{2N}$ for $0 < \rho_2 < \frac{1}{2}$. Also $\frac{1}{3}(2\rho_2 + \eta) < \min(0.5, \delta)$.

$$\begin{aligned} r(\delta) = \max_{\eta,\rho_1,\rho_2} \Big\{ & -H(\frac{4\rho_2 + 2\eta}{3}) + \eta H(\frac{\rho_1}{\eta}) \\ & + (\frac{1}{2} - \eta) H(\frac{\rho_2 - \rho_1}{\frac{1}{2} - \eta}) \\ & + (\frac{1}{2} - \delta + \frac{2\rho_2 + \eta}{3}) H(\frac{\eta/2}{\frac{1}{2} - \delta + \frac{2\rho_2+\eta}{3}}) \\ & + (\delta - \frac{2\rho_2 + \eta}{3}) H(\frac{\eta/2}{\delta - \frac{2\rho_2+\eta}{3}}) \\ & + (\eta + \rho_2 - 2\rho_1) \ln 3 \Big\} \end{aligned} \tag{23}$$

where $H(\cdot)$ is the binary (natural) entropy function, $H(x) = -x \ln x - (1-x)\ln(1-x)$. Using (23) in (4) we can obtain the minimum E_b/N_o threshold for punctured RA (q=3, p=3). See the next section for results.

For systematic punctured RA (q=4,p=4)

$$A_{w,d}^{rep(4)-acc(4)} = \frac{\binom{N}{w}}{\binom{4N}{4w}} \sum_{h=0}^{N} \sum_{j=0}^{N-h} \sum_{i=0}^{2N-h-2j} \binom{2N-h-2j}{i} \binom{N-h}{j} \times \binom{N-d+w}{\lfloor \frac{h}{2} \rfloor} \binom{d-w-1}{\lceil \frac{h}{2} \rceil - 1} \times 2^{2h+2j} \delta_{4w,h+2i+2j} \tag{24}$$

Now we obtain the asymptotic expression of $r(\delta)$ for punctured RA (q=4,p=4), after summing (24) over w. Let $\delta \triangleq \frac{d}{2N}$ for $0 < \delta < 1$, $\eta \triangleq \frac{h}{2N}$ for $0 < \eta < 1/2$, $\rho_1 \triangleq \frac{i}{2N}$ for $0 < \rho_1 < 1 - \eta - 2\rho_2$, and $\rho_2 \triangleq \frac{j}{2N}$ for $0 < \rho_2 < \frac{1}{2} - \eta$. Also $\frac{1}{4}(\eta + 2\rho_1 + 2\rho_2) < \min(0.5, \delta)$.

$$\begin{aligned} r(\delta) = \max_{\eta,\rho_1,\rho_2} \Big\{ & -\frac{3}{2} H(\frac{\eta + 2\rho_1 + 2\rho_2}{2}) \\ & + (1 - \eta - 2\rho_2) H(\frac{\rho_1}{1 - \eta - 2\rho_2}) \\ & + (\frac{1}{2} - \eta) H(\frac{\rho_2}{\frac{1}{2} - \eta}) \\ & + (\frac{1}{2} - \delta + \frac{\eta + 2\rho_1 + 2\rho_2}{4}) H(\frac{\eta/2}{\frac{1}{2} - \delta + \frac{\eta+2\rho_1+2\rho_2}{4}}) \\ & + (\delta - \frac{\eta + 2\rho_1 + 2\rho_2}{4}) H(\frac{\eta/2}{\delta - \frac{\eta+2\rho_1+2\rho_2}{4}}) \\ & + (2\eta + 2\rho_2) \ln 2 \Big\} \end{aligned} \tag{25}$$

Using (25) in (4) we can obtain the minimum E_b/N_o threshold for punctured RA (q=4, p=4). See the next section for results.

IV. ML BER Performance of RA codes with Regular Puncturing

RA codes are usually non-systematic codes, i.e. the information block is not sent along with the output of the accumulator. However, the RA codes with puncturing should be systematic in order to be decodable by iterative decoding. Fig. 5 illustrates the normalized distance spectrum of rate 1/2 codes for a block size of 4000. These codes are RA code (q=2) which will be denoted by RA(2), systematic RA codes with puncturing for (q=3,p=3) which will be denoted by RA(3,3), and (q=4,p=4) which will be denoted by RA(4,4). Also for comparison the normalized distance spectrum of rate 1/2 random codes are shown in Fig. 5. The E_b/N_o thresholds of these codes for infinite block lengths using (4) [7] [8] are compared in Table I. Discrepancy between Random code threshold and Shannon limit is due to the upper bound which is slightly loose for rate 1/2. The BER performance of RA(3,3) and RA(4,4) are shown in Fig. 7 and Fig. 8 respectively.

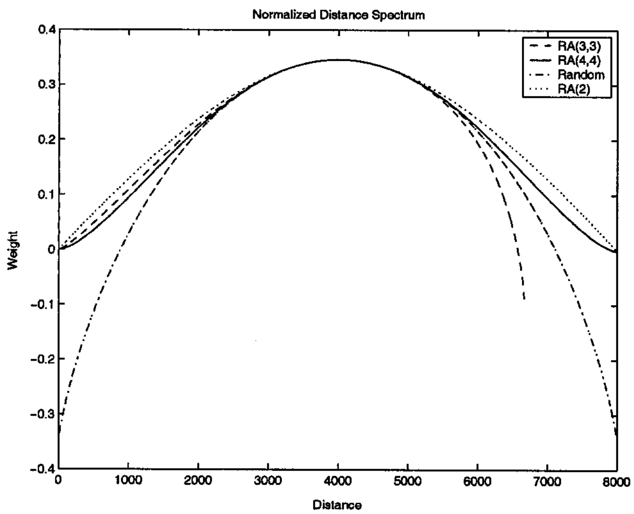

Fig. 5. Normalized distance spectrum $r(\delta)$ vs d of RA codes with puncturing

The asymptotic expression of $r(\delta)$ for RA code with repetition q can be obtained as [4]:

$$r(\delta) = \max_{0<\epsilon<\frac{1}{q}} \left[\frac{1-q}{q}H(q\epsilon) + (1-\delta)H(\frac{q\epsilon}{2(1-\delta)}) + \delta H(\frac{q\epsilon}{2\delta})\right] \tag{26}$$

and the asymptotic expression of $r(\delta)$ for random codes with code rate R_c is $r(\delta) = H(\delta) + (R_c - 1)\ln 2$.

Rate 1/2	RA(q=2)	RA_punc (q=3,p=3)	RA_punc (q=4,p=4)	ARA_punc (q=3,p=3)	ARA_punc (q=4,p=4)	Random Code	Shannon limit
Eb/No threshold	3.364 dB	1.497 dB	0.871 dB	0.509 dB	0.31 dB	0.308 dB	0.187 dB

TABLE I

V. ML Performance of ARA codes

In this section we obtain the ML performance of ARA codes, as a precoded RA code with puncturing, using an accumulator as a precoder. In ARA codes, a portion of the information block goes to the accumulator. In other words, M bits are passed through without any change and the rest (N-M bits) goes through an accumulator. Then the overall output bits are applied to the punctured RA code. The use of these M bits is essential for the iterative decoding to start the message-passing algorithm. M is considered a parameter in code design. The effect of this parameter is studied in ML decoding. Fig. 6 (a) shows a block diagram of the precoder. Fig. 6 (b) shows the ARA code.

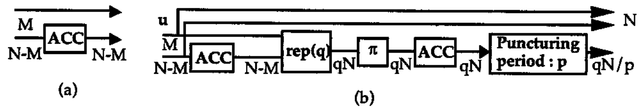

Fig. 6. The block diagram of the precoder (b) ARA code

In order to find the performance of the code we need to compute the IOWE of the precoder. It is easily computed using the IOWE of the accumulator code as follows:

$$A^{pre}_{w,d} = \sum_{m=0}^{M} \binom{M}{m} A^{acc}_{w-m,d-m} \tag{27}$$

Therefore the IOWE of the overall systematic ARA (q,p) code can be written as:

$$A^{pre-rep(q)-acc(p)}_{w,d} = \sum_{l=0}^{N} \frac{A^{pre}_{w,l} A^{rep(q)-acc(p)}_{l,d+l-w}}{\binom{N}{l}} \tag{28}$$

Note that with or without any interleaver between accumulator (precoder) and repetition the same expression holds.

For systematic ARA (q=3,p=3)

$$\begin{aligned} A^{pre-rep(3)-acc(3)}_{w,d} = & \sum_{m=0}^{M}\sum_{l=0}^{N}\sum_{h=0}^{N}\sum_{j=0}^{N}\sum_{i=\max(0,j-N+h)}^{\min(j,h)} \\ & \frac{\binom{M}{m}}{\binom{3N}{3l}}\binom{h}{i}\binom{N-h}{j-i} \\ & \times\binom{N-d+w}{\lfloor h/2\rfloor}\binom{d-w-1}{\lceil h/2\rceil-1} \\ & \times\binom{N-M-l+m}{\lfloor (w-m)/2\rfloor} \\ & \times\binom{l-m-1}{\lceil (w-m)/2\rceil-1} \\ & \times 3^{h+j-2i}\delta_{3l,2j+h} \end{aligned} \tag{29}$$

Now we obtain the asymptotic expression of $r(\delta)$ for ARA (q=3,p=3), using (29). Let $\alpha \triangleq \frac{M}{2N}$ for $0<\alpha<1/2$; $\epsilon_1 \triangleq \frac{M}{2N}$ for $0<\epsilon_1<\alpha$, $\epsilon_2 \triangleq \frac{w-m}{2N}$ for $0<\epsilon_2<\frac{1}{2}-\alpha$, $\delta \triangleq \frac{d}{2N}$ for $0<\delta<1$, $\eta \triangleq \frac{h}{2N}$ for $0<\eta<1/2$, $\rho_1 \triangleq \frac{i}{2N}$ for $\max(0,\rho_2+\eta-\frac{1}{2}) < \rho_1 < \min(\rho_2,\eta)$, and $\rho_2 \triangleq \frac{j}{2N}$ for $0<\rho_2<\frac{1}{2}$.

$$\begin{aligned} r(\delta) = & \max_{\epsilon_1,\epsilon_2,\eta,\rho_1,\rho_2} \Big\{\alpha H(\frac{\epsilon_1}{\alpha}) - \frac{3}{2}H(\frac{4\rho_2+2\eta}{3}) + \eta H(\frac{\rho_1}{\eta}) \\ & +(\frac{1}{2}-\eta)H(\frac{\rho_2-\rho_1}{\frac{1}{2}-\eta}) \\ & +(\frac{1}{2}-\delta+\epsilon_1+\epsilon_2)H(\frac{\eta/2}{\frac{1}{2}-\delta+\epsilon_1+\epsilon_2}) \end{aligned}$$

$$
\begin{aligned}
&+(\delta-\epsilon_1-\epsilon_2)H(\frac{\eta/2}{\delta-\epsilon_1-\epsilon_2}) \\
&+(\frac{1}{2}-\alpha-\frac{1}{3}(2\rho_2+\eta)+\epsilon_1) \\
&\times H(\frac{\epsilon_2/2}{\frac{1}{2}-\alpha-\frac{1}{3}(2\rho_2+\eta)+\epsilon_1}) \\
&+(\frac{1}{3}(2\rho_2+\eta)-\epsilon_1)H(\frac{\epsilon_2/2}{\frac{1}{3}(2\rho_2+\eta)-\epsilon_1}) \\
&+(\eta+\rho_2-2\rho_1)\ln 3\} \qquad (30)
\end{aligned}
$$

Using (30) in (4) we can obtain the minimum E_b/N_o threshold for ARA (q=3, p=3). See the next section for results.

For systematic ARA (q=4,p=4)

$$
\begin{aligned}
A_{w,d}^{pre-rep(4)-acc(4)} = & \sum_{m=0}^{M}\sum_{l=0}^{N}\sum_{h=0}^{N}\sum_{j=0}^{N-h}\sum_{i=0}^{2N-h-2j}\frac{\binom{M}{m}}{\binom{4N}{4l}} \\
& \times\binom{2N-h-2j}{i}\binom{N-h}{j} \\
& \times\binom{N-d+w}{\lfloor\frac{h}{2}\rfloor}\binom{d-w-1}{\lceil\frac{h}{2}\rceil-1} \\
& \times\binom{N-M-l+m}{\lfloor(w-m)/2\rfloor} \\
& \times\binom{l-m-1}{\lceil(w-m)/2\rceil-1} \\
& \times 2^{2h+2j}\delta_{4l,h+2i+2j} \qquad (31)
\end{aligned}
$$

At this point we obtain the asymptotic expression of $r(\delta)$ for ARA (q=4,p=4), using (31). Let $\alpha \triangleq \frac{M}{2N}$ for $0<\alpha<1/2$; $\epsilon_1 \triangleq \frac{M}{2N}$ for $0<\epsilon_1<\alpha$, $\epsilon_2 \triangleq \frac{w-m}{2N}$ for $0<\epsilon_2<\frac{1}{2}-\alpha$, $\delta \triangleq \frac{d}{2N}$ for $0<\delta<1$, $\eta \triangleq \frac{h}{2N}$ for $0<\eta<1/2$, $\rho_1 \triangleq \frac{i}{2N}$ for $0<\rho_1<1-\eta-2\rho_2$, and $\rho_2 \triangleq \frac{j}{2N}$ for $0<\rho_2<\frac{1}{2}-\eta$.

$$
\begin{aligned}
r(\delta) = & \max_{\epsilon_1,\epsilon_2,\eta,\rho_1,\rho_2}\Big\{\alpha H(\frac{\epsilon_1}{\alpha})-2H(\frac{\eta}{2}+\rho_1+\rho_2) \\
&+(1-\eta-2\rho_2)H(\frac{\rho_1}{1-\eta-2\rho_2}) \\
&+(\frac{1}{2}-\eta)H(\frac{\rho_2}{\frac{1}{2}-\eta}) \\
&+(\frac{1}{2}-\delta+\epsilon_1+\epsilon_2)H(\frac{\eta/2}{\frac{1}{2}-\delta+\epsilon_1+\epsilon_2}) \\
&+(\delta-\epsilon_1-\epsilon_2)H(\frac{\eta/2}{\delta-\epsilon_1-\epsilon_2}) \\
&+(\frac{1}{2}-\alpha-\frac{1}{4}(\eta+2\rho_1+2\rho_2)+\epsilon_1) \\
&\times H(\frac{\epsilon_2/2}{\frac{1}{2}-\alpha-\frac{1}{4}(\eta+2\rho_1+2\rho_2)+\epsilon_1}) \\
&+(\frac{1}{4}(\eta+2\rho_1+2\rho_2)-\epsilon_1) \\
&\times H(\frac{\epsilon_2/2}{\frac{1}{4}(\eta+2\rho_1+2\rho_2)-\epsilon_1}) \\
&+(2\eta+2\rho_2)\ln 2\} \qquad (32)
\end{aligned}
$$

Using (32) in (4) we can obtain the minimum E_b/N_o threshold for ARA (q=4, p=4). See the next section for results.

VI. ML BER PERFORMANCE OF ARA CODES

The BER performance bound [7], [8] for ARA(3,3) and ARA(4,4) for different Ms are compared to that of random code for the same input block size (4000) in Fig. 7 and Fig. 8 respectively. It is observed that the more number of bits accumulates in the precoder, the lower the code threshold becomes. However, the improvement diminishes below a certain point, which is M= 1/5 N for ARA(3,3) and M= 2/5 N for ARA(4,4). It is obvious that when M=N the codes turn into RA with puncturing. It is very interesting that the performance of the ARA(4,4) approaches very closely to that of random codes for the same block size in low Eb/No region. For infinite block size we minimize the threshold expression in (4) with respect to α. It is easy to show that the second derivative of the threshold with respect to α is always negative. Thus for ML decoding, the minimum threshold is achieved when $\alpha \to 0$. However, for all values of $0<\alpha<0.1$ in case of ARA(q=3,p=3), and $0<\alpha<0.2$ in case of ARA (q=4,p=4), very small change in threshold was observed.

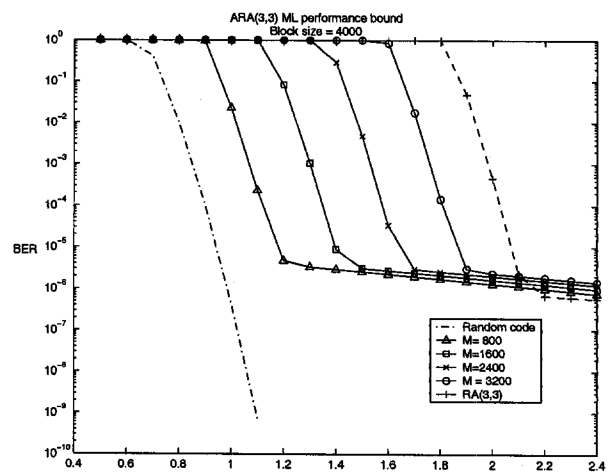

Fig. 7. BER bounds for ARA(3,3) code

It is very instructive to observe the distance spectrum of these codes. As we see in Fig. 9 the only difference between the distance spectrum of these codes and a random code is in the low-distance region, which causes the error floor.

Table I tabulates the minimum Eb/No threshold using (4) for the ARA codes discussed. As we expected based on the BER performance bound for input block of 4000, the E_b/N_o threshold of ARA(4,4) for infinite block is also extremely close to the threshold of random codes. The interleaving gain for for these codes for FER is $1/N$, and for BER is $1/N^2$. Thus precoder improves the SNR threshold but the interleaving gain remains unchanged with respect to RA codes with puncturing.

VII. CONCLUSION

In this paper we analyzed a new channel coding scheme called Accumulate Repeat Accumulate codes (ARA). This

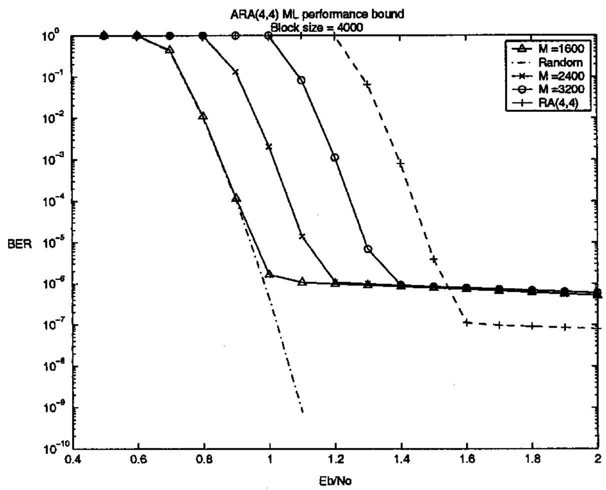

Fig. 8. BER bounds for ARA(4,4) code

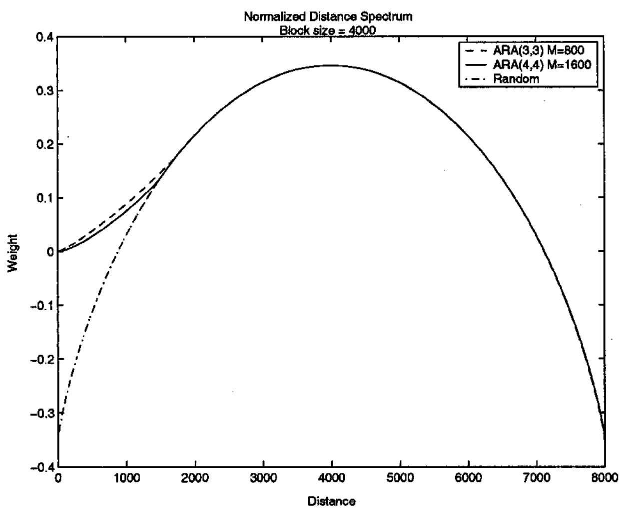

Fig. 9. Normalized distance spectrum $r(\delta)$ vs d of ARA codes

class of codes are a subclass of Low Density Parity Check (LDPC) codes with fast encoder structure. The weight distribution of some simple ARA codes is obtained, and through existing tightest bounds we have shown the ML SNR threshold of ARA codes approaches very closely to the performance of random codes.

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